บทความวิจัย

Boundary Sets, Exterior Sets and Dense Sets in Bi-weak Structure Spaces

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บทคัดย่อ

ในบทความนี้จะนำเสนอแนวคิดของเซตขอบ เซตภายนอก และเซตหนาแน่นในปริภูมิสองโครงสร้างอ่อนได้แสดงให้เห็นสมบัติบางประการของเซตเหล่านี้ โดยเฉพาะอย่างยิ่งได้รับบางลักษณะเฉพาะของเซตในปริภูมิสองโครงสร้างอ่อนโดยใช้เซตขอบ หรือเซตภายนอก

คำสำคัญ : เซตขอบ, เซตหนาแน่น, เซตภายนอก, ปริภูมิสองโครงสร้างอ่อน

Abstract

In this article, the concepts of boundary sets, exterior sets and dense sets in bi-weak structure spaces are introduced. Some properties of their sets are obtained. In particular, some characterizations of closed sets in bi-weak structure spaces using boundary sets or exterior sets are obtained.

Keywords: boundary set, exterior set, dense set, bi-weak structure space.

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Introduction

The notion of bitopological spaces, which consists of a set and two topologies, and some separation axioms in a bitopological space were introduced by Kelly (1963). Later, Popa and Noiri (2000), studied the concept of minimal structures. Csa´szar (2002, 2011), introduced the notions of generalized topologies and weak structures. These structures are generalizations of topologies. Next, the concepts of bigeneralized topological spaces and biminimal structure spaces were introduced by Boonpok (2010). Later, Sompong (2011), studied some properties of boundary sets and exterior sets in biminimal structure spaces, respectively. Moreover, Sompong (2012), studied some properties of dense sets in biminimal structure spaces. Recently, Puiwong et al., (2017), introduced the concept of bi-weak structure spaces or briefly bi-w spaces and studied the properties of closed sets and some separation axioms in bi-weak structure spaces.

In this article, we will extend the concepts of boundary sets, exterior sets and dense sets in bi-weak structure spaces and study some fundamental of their properties.

Methods

In this research, we shall use the methods of proof in mathematics and the basic concepts of weak structures and bi-weak structure spaces. Now, we recall about some properties of weak structures and bi-weak structure spaces.

Definition 2.1. [Csa´szar (2011)]. Let $X$ be a nonempty set and $P(X)$ the power set of $X$. A subfamily $\mathfrak{w}$ of $P(X)$ is called a weak structure (briefly WS) on $X$ if $\emptyset \in \mathfrak{w}$.

By $(X, \mathfrak{w})$ we denote a nonempty set $X$ with a WS $\mathfrak{w}$ on $X$ and it is called a w-space. The elements of $\mathfrak{w}$ are called \textit{w-open sets} and the complements are called \textit{w-closed sets}.

Let $\mathfrak{w}$ be a weak structure on $X$ and $A \subseteq X$, the w-closure of $A$, denoted by $c_\mathfrak{w}(A)$ and w-interior of $A$, denoted by $i_\mathfrak{w}(A)$. We define $c_\mathfrak{w}(A)$ as the intersection of all w-closed sets containing $A$ and $i_\mathfrak{w}(A)$ as the union of all w-open subsets of $A$.

Theorem 2.2. [Csa´szar (2011)]. Let $\mathfrak{w}$ be a WS on $X$ and $A, B \subseteq X$. Then

1. $A \subseteq c_\mathfrak{w}(A)$ and $i_\mathfrak{w}(A) \subseteq A$;
2. If $A \subseteq B$, then $c_\mathfrak{w}(A) \subseteq c_\mathfrak{w}(B)$ and $i_\mathfrak{w}(A) \subseteq i_\mathfrak{w}(B)$;
3. $c_\mathfrak{w}(c_\mathfrak{w}(A)) = c_\mathfrak{w}(A)$ and $i_\mathfrak{w}(i_\mathfrak{w}(A)) = i_\mathfrak{w}(A)$;
4. $c_\mathfrak{w}(X \setminus A) = X \setminus i_\mathfrak{w}(A)$ and $i_\mathfrak{w}(X \setminus A) = X \setminus c_\mathfrak{w}(A)$;
5. $x \in i_\mathfrak{w}(A)$ if and only if there is a w-open set $V$ such that $x \in V \subseteq A$;
6. $x \in c_\mathfrak{w}(A)$ if and only if $V \cap A \neq \emptyset$ for any w-open set $V$ containing;
7. If $A \in \mathfrak{w}$, then $A = i_\mathfrak{w}(A)$. And if $A$ is w-closed, then $A = c_\mathfrak{w}(A)$. 

**Definition 2.3.** [Puiwong et al., (2017)]. Let \( X \) be a nonempty set and \( w^1, w^2 \) be two weak structures on \( X \). A triple \((X, w^1, w^2)\) is called a bi-weak structure space (briefly bi-w space).

Let \((X, w^1, w^2)\) be a bi-w space and \( A \) be a subset of \( X \). The w-closure and w-interior of \( A \) with respect to \( w^i \) are denoted by \( c_{w^i}(A) \) and \( i_{w^i}(A) \) where \( i \in \{1, 2\} \).

**Definition 2.4.** [Puiwong et al., (2017)]. A subset \( A \) of a bi-w space \((X, w^1, w^2)\) is call closed if \( A = c_{w^1}(c_{w^2}(A)) \). The complement of a closed set is called open.

**Remark.** In this paper, we shall call closed and open in a bi-w space that bi-w-closed and bi-w-open, respectively.

**Theorem 2.5.** [Puiwong et al., (2017)]. Let \((X, w^1, w^2)\) be a bi-w space and \( A \) be a subset of \( X \). Then the following are equivalent:

1. \( A \) is bi-w-closed;
2. \( A = c_{w^1}(A) \) and \( A = c_{w^2}(A) \);
3. \( A = c_{w^2}(c_{w^1}(A)) \).

**Proposition 2.6.** [Puiwong et al., (2017)]. Let \((X, w^1, w^2)\) be a bi-w space and \( A \subseteq X \). If \( A \) is both w-closed respect to \( w^1 \) and \( w^2 \), then \( A \) is a bi-w-closed set in the bi-w space \((X, w^1, w^2)\).

**Proposition 2.7.** [Puiwong et al., (2017)]. Let \((X, w^1, w^2)\) be a bi-w space. If \( A_\alpha \) is bi-w-closed for all \( \alpha \in \Lambda \neq \emptyset \), then \( \bigcap_{\alpha \in \Lambda} A_\alpha \) is bi-w-closed.

**Proposition 2.8.** [Puiwong et al., (2017)]. Let \((X, w^1, w^2)\) be a bi-w space. If \( A_\alpha \) is bi-w-open for all \( \alpha \in \Lambda \neq \emptyset \), then \( \bigcup_{\alpha \in \Lambda} A_\alpha \) is bi-w-open.

**Theorem 2.9.** [Puiwong et al., (2017)]. Let \((X, w^1, w^2)\) be a bi-w space and \( A \) be a subset of \( X \). Then the following are equivalent:

1. \( A \) is bi-w-open;
2. \( A = i_{w^1}(i_{w^2}(A)) \);
3. \( A = i_{w^1}(A) \) and \( A = i_{w^2}(A) \);
4. \( A = i_{w^2}(i_{w^1}(A)) \).

**Results**

In this section, we introduce the concepts of boundary sets, exterior sets and dense sets in bi-weak structure spaces and study some fundamental of their properties. Next, let \( i, j \in \{1, 2\} \) be such that \( i \neq j \).

3.1 **Boundary sets in bi-weak structure spaces**

**Definition 3.1.1.** Let \((X, w^1, w^2)\) be a bi-w space, \( A \) be a subset of \( X \) and \( x \in X \). We called \( x \) is a \( w^i w^j \)-boundary point of \( A \) if \( x \in c_{w^i}(c_{w^j}(A)) \) \( \cap \ c_{w^j}(c_{w^i}(X \setminus A)) \). We denote the set of all \( w^i w^j \)-boundary points of \( A \) by \( wBd_{ij}(A) \).
Remark. From the above definition, it is easy to verify that $wBdri_j(A) = c_{w_i}(c_{w_j}(A)) \cap c_{w_j}(X \setminus A)$.

Example 3.1.2. Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset, \{1\}, \{2,3\}\}$ and $w^2 = \{\emptyset, \{3\}, \{1,2\}\}$. Hence $wBdri_2(\{1\}) = X$ and $wBdri_2(\{1\}) = \{1,2\}$.

Lemma 3.1.3. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $wBdri_j(X \setminus A) = wBdri_j(A)$.

Proof. Since $wBdri_j(X \setminus A) = c_{w_i}(c_{w_j}(X \setminus A)) \cap c_{w_j}(c_{w_i}(X \setminus A))$ and $wBdri_j(A) = c_{w_i}(c_{w_j}(A)) \cap c_{w_j}(c_{w_i}(A))$, $wBdri_j(X \setminus A) = wBdri_j(A)$.

Theorem 3.1.4. Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. Then the following statements hold:

1. $wBdri_j(A) = c_{w_i}(c_{w_j}(A)) \setminus i_{w_i}(i_{w_j}(A))$;
2. $wBdri_j(A) \cap i_{w_i}(i_{w_j}(A)) = \emptyset$;
3. $wBdri_j(A) \cap i_{w_i}(i_{w_j}(X \setminus A)) = \emptyset$;
4. $c_{w_i}(c_{w_j}(A)) = wBdri_j(A) \cup i_{w_i}(i_{w_j}(A))$
5. $X = i_{w_i}(i_{w_j}(A)) \cup wBdri_j(A) \cup i_{w_i}(i_{w_j}(X \setminus A))$ is a pairwise disjoint union;
6. $c_{w_i}(c_{w_j}(A)) = wBdri_j(A) \cup A$.

Proof.

1. $wBdri_j(A) = c_{w_i}(c_{w_j}(A)) \cap c_{w_j}(c_{w_i}(X \setminus A))$
   $= c_{w_i}(c_{w_j}(A)) \cap c_{w_j}(X \setminus i_{w_i}(A))$
   $= c_{w_i}(c_{w_j}(A)) \cap X \setminus i_{w_i}(i_{w_j}(A))$
   $= c_{w_i}(c_{w_j}(A)) \setminus i_{w_i}(i_{w_j}(A))$.

2. From 1., we obtain that $wBdri_j(A) \cap i_{w_i}(i_{w_j}(A)) = [c_{w_i}(c_{w_j}(A)) \setminus i_{w_i}(i_{w_j}(A))] \cap i_{w_i}(i_{w_j}(A)) = \emptyset$.

3. $wBdri_j(A) \cap i_{w_i}(i_{w_j}(X \setminus A)) = [c_{w_i}(c_{w_j}(A)) \cap c_{w_j}(c_{w_i}(X \setminus A))] \cap i_{w_i}(i_{w_j}(X \setminus A))$
   $= c_{w_i}(c_{w_j}(A)) \cap c_{w_j}(c_{w_i}(X \setminus A)) \cap (X \setminus c_{w_j}(c_{w_i}(A)))$
   $= \emptyset$.

4. $wBdri_j(A) \cup i_{w_i}(i_{w_j}(A)) = [c_{w_i}(c_{w_j}(A)) \setminus i_{w_i}(i_{w_j}(A))] \cup i_{w_i}(i_{w_j}(A))$
   $= c_{w_i}(c_{w_j}(A)) \cup i_{w_i}(i_{w_j}(A))$
   $= c_{w_i}(c_{w_j}(A))$.

5. $i_{w_i}(i_{w_j}(A)) \cup wBdri_j(A) \cup i_{w_i}(i_{w_j}(X \setminus A)) = c_{w_i}(c_{w_j}(A)) \cup i_{w_i}(i_{w_j}(X \setminus A))$
   $= c_{w_i}(c_{w_j}(A)) \cup i_{w_i}(X \setminus c_{w_j}(A))$
   $= c_{w_i}(c_{w_j}(A)) \cup X \setminus c_{w_j}(c_{w_i}(A))$
   $= X$.

By 2. and 3., we have $wBdri_j(A) \cap i_{w_i}(i_{w_j}(A)) = \emptyset$ and $wBdri_j(A) \cap i_{w_i}(i_{w_j}(X \setminus A)) = \emptyset$. Now, we will show that $i_{w_i}(i_{w_j}(A)) \subseteq A$ and $i_{w_i}(i_{w_j}(X \setminus A)) \subseteq X \setminus A$, we also have
$i_w(i_w(A)) \cap i_w(i_w(X \setminus A)) = \emptyset$. Therefore $X = i_w(i_w(A)) \cup w\text{Bdr}_{ij}(A) \cup i_w(i_w(X \setminus A))$ is a pairwise disjoint union.

6. $w\text{Bdr}_{ij}(A) \cup A = [c_w(c_w(A)) \cap c_w(c_w(X \setminus A))] \cup A$
   $= [c_w(c_w(A)) \cup A] \cap [c_w(c_w(X \setminus A)) \cup A]$
   $= c_w(c_w(A)) \cap [c_w(X \setminus i_w(A)) \cup A]$
   $= c_w(c_w(A)) \cap [X \setminus i_w(i_w(A)) \cup A]$
   $= c_w(c_w(A)) \cap X$
   $= c_w(c_w(A))$.

**Theorem 3.1.5.** Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. Then

1. $A$ is bi-w-closed if and only if $w\text{Bdr}_{ij}(A) \subseteq A$.

2. $A$ is bi-w-open if and only if $w\text{Bdr}_{ij}(A) \subseteq X \setminus A$.

**Proof.** 1. $(\Rightarrow)$ Assume that $A$ is bi-w-closed. Thus $c_w(c_w(A)) = A$, and so $w\text{Bdr}_{ij}(A) \cap (X \setminus A) = c_w(c_w(A)) \cap c_w(c_w(X \setminus A)) \cap (X \setminus A) = \emptyset$. Therefore $w\text{Bdr}_{ij}(A) \subseteq A$.

   $(\Leftarrow)$ Assume that $w\text{Bdr}_{ij}(A) \subseteq A$. Thus $w\text{Bdr}_{ij}(A) \cap (X \setminus A) = \emptyset$, and so $c_w(c_w(A)) \cap c_w(c_w(X \setminus A)) \cap (X \setminus A) = \emptyset$. Since $X \setminus A \subseteq c_w(c_w(X \setminus A))$, we have $c_w(c_w(A)) \cap (X \setminus A) = \emptyset$. Then $c_w(c_w(A)) \subseteq A$. Clearly, $A \subseteq c_w(c_w(A))$. Consequently $A = c_w(c_w(A))$. Hence $A$ is bi-w-closed.

2. $(\Rightarrow)$ Assume that $A$ is bi-w-open. Thus $i_w(i_w(A)) = A$, and so $w\text{Bdr}_{ij}(A) \cap A = c_w(c_w(A)) \cap c_w(c_w(X \setminus A)) \cap A = c_w(c_w(A)) \cap (X \setminus i_w(i_w(A))) \cap A = c_w(c_w(A)) \cap (X \setminus A) = \emptyset$.

Therefore $w\text{Bdr}_{ij}(A) \subseteq X \setminus A$.

$(\Leftarrow)$ Assume that $w\text{Bdr}_{ij}(A) \subseteq X \setminus A$. Thus $w\text{Bdr}_{ij}(A) \cap A = \emptyset$, and so $c_w(c_w(A)) \cap c_w(c_w(X \setminus A)) \cap A = \emptyset$. Then $c_w(c_w(A)) \cap (X \setminus i_w(i_w(A))) \cap A = \emptyset$. Since $A \subseteq c_w(c_w(A))$, we have $(X \setminus i_w(i_w(A))) \cap A = \emptyset$. Thus $A \subseteq i_w(i_w(A))$. Clearly, $i_w(i_w(A)) \subseteq A$. Hence $A = i_w(i_w(A))$. Consequently $A$ is bi-w-open.

**Corollary 3.1.6.** Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $w\text{Bdr}_{ij}(A) = \emptyset$ if and only if $A$ is bi-w-closed and bi-w-open.

**Proof.** $(\Rightarrow)$ Assume that $w\text{Bdr}_{ij}(A) = \emptyset$. Thus, we have $w\text{Bdr}_{ij}(A) \subseteq A$ and $w\text{Bdr}_{ij}(A) \subseteq X \setminus A$.

By Theorem 3.1.5, we have $A$ is bi-w-closed and bi-w-open.

$(\Leftarrow)$ Assume that $A$ is bi-w-closed and bi-w-open. By Theorem 3.1.5, we have $w\text{Bdr}_{ij}(A) \subseteq A$ and $w\text{Bdr}_{ij}(A) \subseteq X \setminus A$. Therefore $w\text{Bdr}_{ij}(A) \subseteq A \cap (X \setminus A) = \emptyset$. Hence $w\text{Bdr}_{ij}(A) = \emptyset$.

### 3.2 Exterior sets in bi-wake structure spaces

**Definition 3.2.1.** Let $(X, w^1, w^2)$ be a bi-w space, $A$ be a subset of $X$ and $x \in X$. We called $x$ is a $w^1w^j$-exterior point of $A$ if $x \in i_w(i_w(X \setminus A))$. We denote the set of all $w^1w^j$-exterior points of $A$ by $w\text{Ext}_{ij}(A)$.

**Remark** From the previous definition, it is easy to verify that $w\text{Ext}_{ij}(A) = X \setminus c_w(c_w(A))$. 

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Example 3.2.2. Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset, \{1\}, \{2,3\}\}$ and $w^2 = \{\emptyset, \{1\}, \{1,2\}\}$ . Hence $w\text{Ext}_{12}(\{2\}) = X\setminus c_{w^1}(c_{w^2}(\{2\})) = \{1\}$ and $w\text{Ext}_{21}(\{2\}) = X\setminus c_{w^2}(c_{w^1}(\{2\})) = \{1\}$ . Moreover, $w\text{Ext}_{12}(\emptyset) = X\setminus c_{w^1}(c_{w^2}(\emptyset)) = \{1\}$ and $w\text{Ext}_{21}(\emptyset) = X\setminus c_{w^2}(c_{w^1}(\emptyset)) = \{2,3\}$.

Lemma 3.2.3. Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. Then

1. $w\text{Ext}_{ij}(A) \cap A = \emptyset$.
2. $w\text{Ext}_{ij}(X) = \emptyset$.

Proof. 1. Since $A \subseteq c_{w^i}(c_{w^j}(A))$, $(X\setminus c_{w^i}(c_{w^j}(A))) \cap A \subseteq (X\setminus A) \cap A = \emptyset$. From $w\text{Ext}_{ij}(A) = X\setminus c_{w^i}(c_{w^j}(A))$, we have $w\text{Ext}_{ij}(A) \cap A = \emptyset$.

2. From 1. and $w\text{Ext}_{ij}(X) \subseteq X$ we have $w\text{Ext}_{ij}(X) = w\text{Ext}_{ij}(X) \cap X = \emptyset$.

Theorem 3.2.4. Let $(X, w^1, w^2)$ be a bi-w space and $A, B$ be two subsets of $X$. If $A \subseteq B$, then $w\text{Ext}_{ij}(B) \subseteq w\text{Ext}_{ij}(A)$.

Proof. Assume that $A \subseteq B$. Then $c_{w^i}(c_{w^j}(A)) \subseteq c_{w^i}(c_{w^j}(B))$ and so $X\setminus c_{w^i}(c_{w^j}(B)) \subseteq X\setminus c_{w^i}(c_{w^j}(A))$.

Hence $w\text{Ext}_{ij}(B) \subseteq w\text{Ext}_{ij}(A)$.

Theorem 3.2.5. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $A$ is bi-w-closed if and only if $w\text{Ext}_{ij}(A) = X\setminus A$.

Proof. $(\Rightarrow)$ Assume that $A$ is bi-w-closed. Then $A = c_{w^i}(c_{w^j}(A))$. Since $w\text{Ext}_{ij}(A) = X\setminus c_{w^i}(c_{w^j}(A)) = X\setminus A$. Therefore $w\text{Ext}_{ij}(A) = X\setminus A$.

$(\Leftarrow)$ Assume that $w\text{Ext}_{ij}(A) = X\setminus A$. Thus $X\setminus c_{w^i}(c_{w^j}(A)) = X\setminus A$. Consequently $c_{w^i}(c_{w^j}(A)) = A$. Hence $A$ is bi-w-closed.

Corollary 3.2.6. Let $(X, w^1, w^2)$ be a bi-w space and $A$ be a subset of $X$. Then $A$ is bi-w-open if and only if $w\text{Ext}_{ij}(X\setminus A) = \emptyset$.

Proof. It follows from Theorem 3.2.5.

Corollary 3.2.7. Let $(X, w^1, w^2)$ be a bi-w space and $A \subseteq X$. If $A$ is bi-w-closed, then $w\text{Ext}_{ij}(X\setminus w\text{Ext}_{ij}(A)) = w\text{Ext}_{ij}(A)$.

Proof. Assume that $A$ is bi-w-closed. From Theorem 3.2.5, $w\text{Ext}_{ij}(A) = X\setminus A$. Then $A = X\setminus w\text{Ext}_{ij}(A)$.

Hence $w\text{Ext}_{ij}(X\setminus w\text{Ext}_{ij}(A)) = w\text{Ext}_{ij}(A)$.

Theorem 3.2.8. Let $(X, w^1, w^2)$ be a bi-w space and $A, B$ be two subsets of $X$. Then;

1. $w\text{Ext}_{ij}(A) \cup w\text{Ext}_{ij}(B) \subseteq w\text{Ext}_{ij}(A \cap B)$.
2. If $A$ and $B$ are bi-w-closed, then $w\text{Ext}_{ij}(A) \cup w\text{Ext}_{ij}(B) = w\text{Ext}_{ij}(A \cap B)$

Proof. 1. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Theorem 3.2.4, we have $w\text{Ext}_{ij}(A) \subseteq w\text{Ext}_{ij}(A \cap B)$ and $w\text{Ext}_{ij}(B) \subseteq w\text{Ext}_{ij}(A \cap B)$. It follows that $w\text{Ext}_{ij}(A) \cup w\text{Ext}_{ij}(B) \subseteq w\text{Ext}_{ij}(A \cap B)$.
2. Assume that $A$ and $B$ are bi-$w$-closed. Then $A \cap B$ is bi-$w$-closed. By Theorem 3.2.5, Thus $wExt_{ij}(A \cap B) = X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) = wExt_{ij}(A) \cup wExt_{ij}(B)$.

**Remark** In Theorem 3.2.8, 2. is not true if $A$ and $B$ are not bi-$w$-closed. We can be seen from the following example.

**Example 3.2.9.** Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset,\{1\},\{2,3\}\}$ and $w^2 = \{\emptyset,\{2\},\{1,3\}\}$. Hence $wExt_{12}(\{1\} \cap \{2\}) = X$ , and $wExt_{12}(\{1\}) = X \setminus c_{w^1}(c_{w^2}(\{1\})) = \emptyset$ and $wExt_{12}(\{2\}) = X \setminus c_{w^1}(c_{w^2}(\{2\})) = \{1\}$. Therefore $wExt_{12}(\{1\}) \cup wExt_{12}(\{2\}) \neq wExt_{12}(\{1\} \cap \{2\})$.

**3.3 Dense sets in bi-weak structure spaces**

**Definition 3.3.1.** Let $(X, w^1, w^2)$ be a bi-$w$ space. A subset $A$ of $X$ is called a $w^i w^j$-dense set in $X$ if $X = c_{w^i}(c_{w^j}(A))$.

**Example 3.3.2.** Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset,\{1,2\},\{1,3\},\{2,3\}\}$ and $w^2 = \{\emptyset,\{1\},\{3\},\{2,3\}\}$. Then $c_{w^1}(c_{w^2}(\{3\})) = X$ and $c_{w^2}(c_{w^1}(\{3\})) = \{2,3\}$. Hence $\{3\}$ is a $w^1 w^2$-dense set of $X$ and $\{3\}$ is not a $w^2 w^1$-dense set of $X$.

**Theorem 3.3.3.** Let $(X, w^1, w^2)$ be a bi-$w$ space and $A$ be a subset of $X$. If $A$ is a $w^i w^j$-dense set in $X$, then for any nonempty bi-$w$-closed subset $F$ of $X$ such that $A \subseteq F$, we have $F = X$.

**Proof.** Suppose that $A$ is a $w^i w^j$-dense set in $X$ and $F$ is a bi-$w$-closed subset of $X$ such that $A \subseteq F$. Since $A$ is a $w^i w^j$-dense set in $X$, $X = c_{w^i}(c_{w^j}(A))$. By assumption, $F$ is bi-$w$-closed and $A \subseteq F$, it follows that $X = c_{w^i}(c_{w^j}(A)) \subseteq c_{w^i}(c_{w^j}(F)) = F$. Hence $F = X$.

**Remark** By Theorem 3.3.3, if $A$ is a $w^i w^j$-dense set in $X$, then only $X$ is bi-$w$-closed in $X$ containing $A$. Moreover, it is not true if $F$ is not bi-$w$-closed. We can be seen from the following example.

**Example 3.3.4.** Let $X = \{1,2,3\}$. Define weak structures $w^1$ and $w^2$ on $X$ as follows: $w^1 = \{\emptyset,\{1\},\{1,3\}\}$ and $w^2 = \{\emptyset,\{1\},\{2\},\{1,3\}\}$. Then $c_{w^1}(c_{w^2}(\{1\})) = X$. Hence $\{1\}$ is a $w^1 w^2$-dense set in $X$, but $\{1\}$ is not a $w^1 w^2$-closed set in $X$.

**Theorem 3.3.5.** Let $(X, w^1, w^2)$ be a bi-$w$ space and $A$ be a subset of $X$. The following are equivalent.

1. If $F$ is a nonempty bi-$w$-closed subset of $X$ such that $A \subseteq F$, then $F = X$.

2. $G \cap A \neq \emptyset$ for any nonempty bi-$w$-open subset $G$ of $X$.

**Proof.** (1. $\Rightarrow$ 2.) Assume that if $F$ is a nonempty bi-$w$-closed subset of $X$ such that $A \subseteq F$, then $F = X$.

Suppose that $G \cap A = \emptyset$ for some nonempty bi-$w$-open subset $G$ of $X$. Thus $A \subseteq X \setminus G$. Since $G$ is bi-$w$-open, $X \setminus G$ is bi-$w$-closed. By assumption, we have $X \setminus G = X$. Therefore $G = \emptyset$, which is contradiction. Hence $G \cap A \neq \emptyset$ for any nonempty bi-$w$-open subset $G$ of $X$.

(2. $\Rightarrow$ 1.) Assume that 2. holds, and $F$ is a nonempty bi-$w$-closed subset of $X$ such that $A \subseteq F$. Suppose that $F \neq X$. Thus $X \setminus F$ is a nonempty bi-$w$-open subset of $X$. By assumption, we have $(X \setminus F) \cap A \neq \emptyset$. This is contradiction with $A \subseteq F$. Therefore $F = X$. 

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Corollary 3.3.6. Let \((X, w^1, w^2)\) be a bi-w space and \(A \subseteq X\). If \(A\) is a \(w^1w^j\)-dense set in \(X\), then \(G \cap A \neq \emptyset\) for any nonempty bi-w-open subset \(G\) of \(X\).

Proof. It follows from Theorem 3.3.3 and Theorem 3.3.5.

Theorem 3.3.7. Let \((X, w^1, w^2)\) be a bi-w space and \(A\) be a subset of \(X\). Then \(i_{w^j}(w^j(X \setminus A)) = \emptyset\) if and only if \(A\) is a \(w^1w^j\)-dense set in \(X\).

Proof. \((\Rightarrow)\) Assume that \(i_{w^j}(w^j(X \setminus A)) = \emptyset\). Then \(X \setminus c_{w^j}(c_{w^j}(A)) = \emptyset\), so \(i_{w^j}(w^j(X \setminus A)) = X \setminus c_{w^j}(c_{w^j}(A)) = \emptyset\).

\((\Leftarrow)\) Suppose that \(A\) is a \(w^1w^j\)-dense set in \(X\). Then we have \(c_{w^j}(c_{w^j}(A)) = X\), and so \(i_{w^j}(w^j(X \setminus A)) = X \setminus c_{w^j}(c_{w^j}(A)) = \emptyset\).

Theorem 3.3.8. Let \((X, w^1, w^2)\) be a bi-w space and \(A\) be a subset of \(X\). Then \(A\) is a \(w^1w^j\)-dense set in \(X\) if and only if \(w\text{Ext}_{ij}(A) = \emptyset\).

Proof. \((\Rightarrow)\) Suppose that \(A\) is a \(w^1w^j\)-dense set in \(X\). Then \(w\text{Ext}_{ij}(A) = X \setminus c_{w^j}(c_{w^j}(A)) = X \setminus X = \emptyset\).

\((\Leftarrow)\) Assume that \(w\text{Ext}_{ij}(A) = \emptyset\). Then \(X \setminus c_{w^j}(c_{w^j}(A)) = \emptyset\). It follows that \(c_{w^j}(c_{w^j}(A)) = X\). Therefore \(A\) is a \(w^1w^j\)-dense set in \(X\).

Discussion

In 2011 and 2012, the notions and properties of boundary sets, exterior sets and dense sets in bi-minimal structure spaces are studied Sompon. In 2017, Puiwong et al. studied bi-weak structure spaces. It is obvious that a bi-minimal structure space is a bi-weak structure space. We investigated the above concepts into bi-weak structure spaces. The similar properties in bi-minimal structure spaces are obtained in bi-weak structure spaces except the property of exterior of empty set, that is, the exterior of empty set is not empty set in bi-weak structure spaces.

Conclusions

In this paper, we introduced and studied boundary sets, exterior sets and dense sets in a bi-weak structure space. We obtained some of their properties. In particular, we gave some characterizations of closed sets in a bi-weak structure space. That is, \(A\) is bi-w-closed if and only if \(w\text{Bd}_{ij}(A) \subseteq A\). And \(A\) is bi-w-closed if and only if \(w\text{Ext}_{ij}(A) = X \setminus A\). Moreover, we also obtained a characterization of dense sets, i.e., \(A\) is a \(w^1w^j\)-dense set in \(X\) if and only if \(w\text{Ext}_{ij}(A) = \emptyset\).
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References


