แผนภูมิควบคุมสู่ขาวรูปแบบการผลิตที่มีการแจกแจงแบบปัวซองนี้จัดทั่วไป ซึ่งมีศูนย์มากกับการกระจายที่มากเกินจริง

Control Charts for Zero-Inflated Generalized Poisson Process with Over-dispersion

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บทคัดย่อ

งานวิจัยนี้มีจุดประสงค์เพื่ศึกษาแผนภูมิควบคุมสู่ขาวรูปแบบการเปลี่ยนแปลงต่างๆ จำนวนรอยตัวหนึ่ง (λ) เมื่อกระบวนการผลิตมีการแจกแจงแบบปัวซองนี้จัดทั่วไปที่มีศูนย์มาก (ZIGP) ซึ่งมีการกระจายที่มากเกินจริง แผนภูมิควบคุมค่าเฉลี่ยรวมผลรวมซึ่งมีการกระจายที่มากเกินจริงมีค่าสtatิสติกซัมมี่ (CUSUM chart) แผนภูมิควบคุมค่าเฉลี่ยที่มีการกระจายที่มากเกินจริงมีค่าสtatิสติกซัมมี่ซัมมี่ (EWMAZ chart) และแผนภูมิควบคุมจำนวนรอยตัวหนึ่งต่อหน่วยเพื่อนำมาเปรียบเทียบประสิทธิภาพของแผนภูมิควบคุมคุณภาพที่มีการเปลี่ยนแปลงในค่า λ พบว่า EWMAZ chart มีประสิทธิภาพในการตัดสินใจแสดงตัวเลขสถิติที่มีการเปลี่ยนแปลงในค่า λ ที่มีการกระจายที่มากเกินจริง (φ) ต่อหน่วย

คำสำคัญ : การกระจายที่มากเกินจริง, การแจกแจงแบบปัวซองนี้จัดทั่วไปที่มีศูนย์มาก, แผนภูมิควบคุมค่าเฉลี่ยต่อหน่วยต่อหน่วย, แผนภูมิควบคุมค่าเฉลี่ยรวมผลรวม, ประสิทธิภาพ
Abstract

This paper aims to study charts for detecting the mean of nonconformities ($\lambda$) shifts based on the zero-inflated generalize poisson (ZIGP) process with over-dispersion. The first chart is the same as a CUSUM-chart where CUSUM statistics constructed base on log-likelihood ratio called, $\lambda -$ CUSUM chart, the second chart is the same as a EWMA-chart based on ZIGP process called, EWMA$_{r}$-chart, the third chart is the same as a c-chart based on ZIGP process called, $c_{r}$-chart. The performance of control the charts was considered from the average run length. The research result shows that for the $\lambda$ shifts process, the EWMA$_{r}$-chart is the best for all level of $\lambda$, proportion zero ($\omega$), mean shift ($\rho$) and over-dispersion ($\varphi$).

Keywords: over-dispersion, zero-inflated generalized poisson distribution, The exponentially weighted moving average control chart

Introduction

Traditional poisson distribution is inappropriate for an excess number of zero nonconformities in processes. This paper focuses on zero-inflated generalize poisson (ZIGP) distribution. The ZIGP developed from generalized poisson distribution (GPD) has two parameters $\lambda$ and $\varphi$, where $\lambda$ is the mean of the nonconformities in a sample unit and $\varphi$ is the over-dispersion (Famoye and Singh, 2003). ZIGP distribution is the generalized poisson distribution combined with a mixture of the proportion of zero nonconformities ($\omega$) (Famoye and Singh, 2006).

The $c$-chart, base on the poisson distribution is used to monitor nonconformities in processes. However, if there is an excess number of zero nonconformities in the processes then the $c$-chart is an unsuitable control chart. This paper is interested in the cumulative sum chart (CUSUM-chart) and the exponentially weighted moving average (EWMA-chart). Because the CUSUM-chart and the EWMA-chart are efficient charts to monitor smaller shifts in the process, Montgomery, 2005. Page, 1954 first proposed the CUSUM-chart and other authors, Gan, 1990 and Lucas, 1976. Kateme and Mayureesawan, 2013 constructed the CUSUM-chart for the ZIGP process. They show, that the $\omega$ - CUSUM chart, $\lambda$ - CUSUM chart and $\varphi$ - CUSUM chart, are performed for detecting the mean shift of individual parameter. Gan, 1991 studied the EWMA-chart for detecting the $\lambda$ of the Poisson process. He founded that the EWMA-chart was the best for detecting mean shift in $\lambda$.

The aim of this paper is to study the influence of CUSUM-chart, EWMA-chart and c-chart based on the ZIGP process, in the case where the variance is greater than the mean, called over-dispersion. All of these three charts are used to detect changes in individual parameters of the mean of nonconformities ($\lambda$). The measure of
control chart performance was considered from the average run length (ARL). The \textit{CUSUM-chart} and the \textit{EWMA-chart} based on the ZIGP process were compared with the \textit{c-chart} based on the ZIGP distribution.

\textbf{Methods}

\textbf{The Zero-Inflated Generalized Poisson (ZIGP) distribution}

The probability function is given by: (Famoye and Singh, 2006)

\begin{equation}
    p(Y = y) = \begin{cases}
        \omega + (1 - \omega) \exp(-\lambda \phi), & y = 0 \\
        \exp\left(-\frac{1}{\phi}(\lambda + y(\phi - 1))\right) \frac{\lambda^y (\lambda + y(\phi - 1))^{y-1}}{\phi^y y!}, & y > 0
    \end{cases}
\end{equation}

where \( y \) is the random variables of nonconformities in a sample unit,
\( \lambda \) is the mean of nonconformities in a sample unit based on the ZIGP distribution,
\( \omega \) is a measure of the extra proportion of zero nonconformity in a sample unit,
\( \phi \) is the over-dispersion for ZIGP distribution, and

\begin{align}
    E(Y) &= (1 - \omega)\lambda \quad \text{and} \quad V(Y) = (1 - \omega)\lambda (\phi^2 + \lambda \omega).
\end{align}

\textbf{The Cumulative Sum Chart based on a ZIGP distribution (\( \lambda_z \) - CUSUM chart)}

1. The \( \lambda_z \) - \textit{CUSUM chart} is a \textit{CUSUM chart} for detecting shifts in a parameter \( \lambda \). The cumulative sum statistics constructed base on log-likelihood ratio for plotting on the \( \lambda_z \) - \textit{CUSUM chart} (\( A_z \)) defined as: (Katemee and Mayureesawhan, 2013)

\begin{equation}
    A_z = \max\left(0, A_{z-1} + D_z \right), \quad z = 1, 2, \ldots
\end{equation}

The head start value of the cumulative sum statistics (\( A_0 \)) is 0 and \( D_z \) is the log-likelihood ratio of ZIGP distribution for a shift in parameter \( \lambda \) (\( \lambda_z \)) defined as follows:
where $y_i$ is the observations of $y$ taken at the time $i$,

$\lambda_z$ is the in-control value of the mean number of nonconformities for ZIGP distribution,

$\lambda_i$ is the out-of-control values of the mean number of nonconformities for ZIGP distribution,

$\omega_z$ is the in-control value of the proportion of zero nonconformity for ZIGP distribution,

$\phi_z$ is the in-control value of the over dispersion for ZIGP distribution,

The $Z_i$ - CUSUM chart will signals in the process when $Z_i > H_i$, where $H_i$ is the UCL of the $Z_i$ - CUSUM chart that is determined based on required in-control performance.

The Exponentially Weighted Moving Average chart is based on ZIGP distribution ($EWMA_z$-chart)

The $EWMA_z$-chart is the same as a $EWMA$-chart based on detection of changes in parameter $\lambda$ of the ZIGP distribution. The $EWMA$ statistics for plotting on the $EWMA_z$-chart ($Z_i$) are defined as

$$Z_i = \xi y_i + (1-\xi)Z_{i-1}, \quad i = 1,2,...$$

The head start value of the $EWMA$ statistics($Z_0$) = $\lambda_z$

where $\xi$ = a constant that determinations must satisfy $0 < \xi \leq 1$,

$y_i$ is the observation of $y$ taken at time $i$,

$\lambda_z$ is the in-control value of the mean number of nonconformities for ZIGP distribution,

The $EWMA_z$-chart will signal in the process when $Z_i > H_{EWMA}$, where $H_{EWMA}$ is the UCL of the $EWMA_z$-chart that is determined based on required in-control performance.

The Shewhart control chart of nonconformities is based on ZIGP distribution ($c_z$-chart)

The $c_z$-chart is the same as a $c$-chart based on detecting shifts in parameters of the ZIGP distribution. The upper control limit (UCL) of the $c_z$-chart is $c + L \sqrt{c}$, where $c$ is assumed to be the mean number of nonconformities if the mean of the probability distribution is known and $L$ is the coefficient of control limit of
c-chart. The $c_{2}$-chart will signals when any observations of nonconformities (y) is greater than $H_{c}$, where $H_{c}$ is the UCL of the $c_{2}$-chart that is selected matching the desired in-control performance.

Simulation Results

In the simulations where the average run length (ARL) was close to the target, the mean number of nonconformities of: $(\lambda, \rho) = 2.0$ and $4.0$ was set. The proportions of zero nonconformity were: $(\omega, \phi) = 0.3$ and $0.5$. The over dispersions were: $(\lambda, \phi) = 1.2$ and $1.4$. The out-of-control values of the mean number of nonconformities were: $\lambda = \lambda_{o} + \rho$, where the mean shifts were: $(\rho) = 2, 3, 4, 5, 6, 7$ and $8$. The ARL was the criteria for evaluating the performance of the control charts. The seven steps of the research process were:

1. The R program was used to simulate the number of nonconforming items for a ZIGP where the parameters were $(n, \lambda, \phi, \omega)$.  

2. The three value of upper control limit were: the value of $H_{c}$ for $\lambda_{o}$ - CUSUM chart, the value of $H_{\text{EWMA}}$ for $EWMA_{2}$-chart and value of $H_{c}$ for $c_{2}$-chart. Each value of $\lambda, \omega$, and $\phi$ was set in the study. However, the values of $H_{\lambda}, H_{\text{EWMA}}$, and $H_{c}$ were changed to match $ARL_{u} = 370$ for all charts. Calculations, based on 100,000 replications, for the average upper control limit that were in each the parameters.

3. For the $\lambda_{o}$ - CUSUM chart calculations of the log-likelihood ratios for the cumulative sum statistics ($D_{i}$) were plotted in the $A_{i}$ value from (3). The $EWMA_{2}$-chart, calculated the $Z_{i}$ value from (5) were used for the $EWMA_{2}$-chart. The $c_{2}$-chart, simulation of the numbers of nonconforming (y) were used for the $c_{2}$-chart.

4. The $\lambda_{o}$ - CUSUM chart, investigated the $A_{i}$ value with $H_{\lambda}$ to find the run length (RL). The $EWMA_{2}$-chart, investigated the $Z_{i}$ value with $H_{\text{EWMA}}$ value. The $c_{2}$-chart, investigated the $y_{i}$ value with $H_{c}$ value. Consider investigating the $A_{i}, Z_{i}$ and $y_{i}$ that are out-of-control points. When there were points outside the control limit, then they were stored in the observations before a point indicated an out-of-control for the run length (RL) calculation. If they were at $i$ statistics indicated an out-of-control then $RL = i - 1$.

5. One hundred thousand (100,000) replications for the average run length (ARL) were computed form Steps 3 to 4 for each of the charts.

6. When contrasted the performance of control charts that gave a low ARL, then the control charts were determined efficient.

7. Changing during a parameters value in the study to completely.

Results and Discussion

An overview of the control charts shows they were proficient for all levels of the parameters and all levels of shifts. Table 1 defines the levels for parameter shifts when $\lambda = 2, \omega = 0.3$ and $\phi = 1.2$. 

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Table 1 defines the levels for parameter shifts when $\lambda = 2, \omega = 0.3$ and $\phi = 1.2$. 

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Table 2 shows the upper control limit of \( \lambda \)-CUSUM chart (\( H_\lambda \)), EWMA-\( \lambda \)-chart (\( H_{\text{EWMA}} \)) and \( c \)-chart (\( H_c \)) matched with the \( ARL_0 = 370 \). It can be seen that all values of \( \lambda \), \( \phi \) and \( \omega \). If \( H_\lambda \) increased then \( \omega \) was decreasing. However, \( H_{\text{EWMA}} \) and \( \omega \) were going in the same direction. The results showed that the \( c \)-chart returned unsuitable values of \( ARL_0 \), therefore this paper is not shown.

Table 3 and Fig. 1 show the \( ARL_1 \) of \( \lambda \)-CUSUM chart and EWMA-\( \lambda \)-chart for shift in parameter \( \lambda \). The results found that for all \( \lambda \), \( \phi \) and level of a shift, the EWMA-\( \lambda \)-chart returned low values of \( ARL_1 \). That the EWMA-\( \lambda \)-chart gave a better performance than \( \lambda \)-CUSUM chart because it was able to detect the shift faster.

**Table 1** Defines the levels for parameter shifts in \( \lambda \) when \( \lambda = 2 \)

<table>
<thead>
<tr>
<th>Levels of Shifts</th>
<th>1 ( \lambda = 4 )</th>
<th>2 ( \lambda = 5 )</th>
<th>3 ( \lambda = 6 )</th>
<th>4 ( \lambda = 7 )</th>
<th>5 ( \lambda = 8 )</th>
<th>6 ( \lambda = 9 )</th>
<th>7 ( \lambda = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter shifts in ( \lambda ) (( \lambda = \lambda + \rho ))</td>
<td>( \lambda = 4 )</td>
<td>( \lambda = 5 )</td>
<td>( \lambda = 6 )</td>
<td>( \lambda = 7 )</td>
<td>( \lambda = 8 )</td>
<td>( \lambda = 9 )</td>
<td>( \lambda = 10 )</td>
</tr>
</tbody>
</table>

**Table 2** The upper control limit \( H_\lambda \), \( H_{\text{EWMA}} \) and \( H_c \) were matching with the desired in-control performance for all levels of the \( \lambda \), \( \omega \) and \( \phi \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>4</th>
<th>4</th>
<th>4</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>0.30</td>
<td>0.50</td>
<td>0.30</td>
<td>0.50</td>
<td>0.30</td>
<td>0.50</td>
<td>0.30</td>
<td>0.50</td>
</tr>
<tr>
<td>( \phi )</td>
<td>1.2</td>
<td>1.2</td>
<td>1.4</td>
<td>1.4</td>
<td>1.2</td>
<td>1.2</td>
<td>1.4</td>
<td>1.4</td>
</tr>
<tr>
<td>( H_\lambda )</td>
<td>3.50</td>
<td>3.20</td>
<td>4.68</td>
<td>4.49</td>
<td>3.00</td>
<td>2.70</td>
<td>3.31</td>
<td>2.90</td>
</tr>
<tr>
<td>( ARL_{q0} )</td>
<td>370.3</td>
<td>371.5</td>
<td>369.7</td>
<td>366.9</td>
<td>369.2</td>
<td>369.2</td>
<td>372.8</td>
<td>372.7</td>
</tr>
<tr>
<td>( H_c )</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>9</td>
<td>12</td>
<td>11</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>( ARL_{q0} )</td>
<td>386.2</td>
<td>482.1</td>
<td>489.6</td>
<td>316.2</td>
<td>397.4</td>
<td>290.8</td>
<td>338.6</td>
<td>439.7</td>
</tr>
<tr>
<td>( H_{\text{EWMA}} )</td>
<td>5.155</td>
<td>5.5</td>
<td>6.107</td>
<td>6.573</td>
<td>8.038</td>
<td>8.252</td>
<td>9.15</td>
<td>9.48</td>
</tr>
<tr>
<td>( ARL_{q0} )</td>
<td>370.1</td>
<td>370.4</td>
<td>370.1</td>
<td>370.8</td>
<td>370.7</td>
<td>370.0</td>
<td>370.7</td>
<td>370.3</td>
</tr>
</tbody>
</table>
### Table 3

The ARL of $\lambda_z$ - CUSUM chart and EWMA$_{\phi}$ chart for shift in parameter $\lambda$.

<table>
<thead>
<tr>
<th>$\lambda_z$</th>
<th>$\rho$</th>
<th>$\phi = 1.2$</th>
<th>$\omega = 0.3$</th>
<th>$\omega = 0.5$</th>
<th>$\phi = 1.4$</th>
<th>$\omega = 0.3$</th>
<th>$\omega = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_z$ - CUSUM</td>
<td>$\xi = 0.5$</td>
<td>31.647</td>
<td>16.740</td>
<td>38.757</td>
<td>24.131</td>
<td>47.147</td>
<td>27.834</td>
</tr>
<tr>
<td>EWMA$_{\phi}$</td>
<td>$\xi = 0.613$</td>
<td>12.777</td>
<td>7.487</td>
<td>17.121</td>
<td>11.063</td>
<td>18.651</td>
<td>12.906</td>
</tr>
<tr>
<td>$\lambda_z$ - CUSUM</td>
<td>$\xi = 0.5$</td>
<td>4.113</td>
<td>2.837</td>
<td>7.635</td>
<td>4.135</td>
<td>7.826</td>
<td>4.640</td>
</tr>
<tr>
<td>EWMA$_{\phi}$</td>
<td>$\xi = 0.613$</td>
<td>3.189</td>
<td>2.034</td>
<td>6.042</td>
<td>3.010</td>
<td>5.965</td>
<td>3.220</td>
</tr>
<tr>
<td>$\lambda_z$ - CUSUM</td>
<td>$\xi = 0.5$</td>
<td>4.159</td>
<td>3.459</td>
<td>7.884</td>
<td>5.170</td>
<td>7.827</td>
<td>5.035</td>
</tr>
<tr>
<td>EWMA$_{\phi}$</td>
<td>$\xi = 0.613$</td>
<td>4.891</td>
<td>2.806</td>
<td>7.884</td>
<td>5.170</td>
<td>8.231</td>
<td>4.389</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda_z$</th>
<th>$\rho$</th>
<th>$\phi = 1.2$</th>
<th>$\omega = 0.3$</th>
<th>$\omega = 0.5$</th>
<th>$\phi = 1.4$</th>
<th>$\omega = 0.3$</th>
<th>$\omega = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_z$ - CUSUM</td>
<td>$\xi = 0.6$</td>
<td>43.49</td>
<td>30.056</td>
<td>53.792</td>
<td>41.106</td>
<td>57.515</td>
<td>45.086</td>
</tr>
<tr>
<td>$\lambda_z$ - CUSUM</td>
<td>$\xi = 0.5$</td>
<td>10.088</td>
<td>7.757</td>
<td>16.622</td>
<td>11.069</td>
<td>13.908</td>
<td>12.270</td>
</tr>
<tr>
<td>EWMA$_{\phi}$</td>
<td>$\xi = 0.613$</td>
<td>7.574</td>
<td>4.891</td>
<td>9.962</td>
<td>7.235</td>
<td>10.037</td>
<td>7.725</td>
</tr>
<tr>
<td>$\lambda_z$ - CUSUM</td>
<td>$\xi = 0.5$</td>
<td>5.670</td>
<td>3.459</td>
<td>7.884</td>
<td>5.170</td>
<td>7.827</td>
<td>5.035</td>
</tr>
<tr>
<td>EWMA$_{\phi}$</td>
<td>$\xi = 0.613$</td>
<td>4.891</td>
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<td>5.170</td>
<td>8.231</td>
<td>4.389</td>
</tr>
</tbody>
</table>
Figure 1  The $ARL_1$ of the, $\lambda_z$ - CUSUM chart and EWMA-$\lambda$-chart for the shifts in parameter $\lambda$.

Conclusions
The purpose of this paper was to study of the CUSUM chart, EWMA-chart and c-chart to detect parameter shifts of $\lambda$ for the ZIGP process with over-dispersion. The three charts were, the $\lambda_z$ - CUSUM chart, EWMA-$\lambda$-chart and c-$\lambda$-chart. The $\lambda_z$ - CUSUM chart was a CUSUM chart where cumulative sum statistics constructed base on the log-likelihood ratio. The EWMA-$\lambda$-chart was a EWMA-chart based on the ZIGP process. The c-$\lambda$-chart was a c-chart based on the ZIGP process. The average run length ($ARL$) of these charts were considered. The results of the comparisons are summarized as follows, the EWMA-$\lambda$-chart performs better than the $\lambda_z$ - CUSUM chart because its performance in detection in mean shift is faster for all level values of the parameters in the processes.

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References


